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ABSTRACT

In this paper, we introduce an abstract KAM (Kolmogorov–Arnold–Moser) theorem. As an application, we study the two-dimensional completely resonant beam system under periodic boundary conditions. Using the KAM theorem together with partial Birkhoff normal form method, we obtain a family of Whitney smooth small–amplitude quasi–periodic solutions for the equation system.

I. INTRODUCTION AND MAIN RESULT

Many partial differential equations (PDEs) arising in physics can be seen as infinit dimensional Hamiltonian systems. In 1998's ICM, Kuksin stated that we do not know what happens to invariant tori of an n-dimensional (in space) linear Schrödinger equation with $n \geq 3$ and of a linear wave equation with $n \geq 2$ under Hamiltonian perturbation. The dynamics of linear Hamiltonian PDEs is quite clear. The general problem discussed here is the persistency of quasi-periodic solutions of linear or integrable equations after Hamiltonian perturbation.

Bourgain[1,2] proved the existence of quasi–periodic solutions for space multidimensional Schrödinger and wave equations,which gave an affirmativ answer to Kuksin's open problem. Bourgain made use of Lyapunov–Schmidt procedure and a Newtonian scheme developed by Craig, Wayne, Bourgain (CWB method for simplicity).[1–9] The scheme of CWB avoids the cumbersome second Melnikov conditions by solving variable–coefficien homological equations. The method is less Hamiltonian and more flexibl than the Kolmogorov–Arnold–Moser (KAM) scheme to deal with resonant cases. This approach is particularly inspiring for PDEs in higher space dimension but to a high cost: the approximate linear equations are variable (quasi-periodic in time) coefficients The disadvantage of CWB method is that one knows nothing on the dynamics around constructed quasi-periodic solutions.

Constructing quasi-periodic solutions of higher dimensional Hamiltonian PDEs by method developed from the finit dimensional KAM theory appeared later.[10–34] The advantage of the method from the finit dimensional KAM theory is the construction of a local normal form in a neighborhood of the obtained solutions in addition to the existence of quasi-periodic solutions. The Birkhoff normal form analysis implies that the frequencies of the expected quasi-periodic solutions vary with their "amplitudes," allowing to prove that the Melnikov non-resonance conditions are satisfie for most amplitudes. The nice normal form is not only an important outcome of the KAM theory, but also a very important ingredient in the proof. The normal form is helpful to understand the dynamics of the corresponding equations. For example, one sees the linear stability and zero Lyapunov exponents. All those methods are well developed for one dimensional Hamiltonian PDEs, however, they meet difficultie in higher dimensional Hamiltonian PDEs. A satisfactory future is under construction.

Geng–You[17,18] proved that the higher dimensional nonlinear beam equations and nonlocal Schrödinger equations admit small–amplitude linearly–stable quasi–periodic solutions. By exploiting the Töplitz–Lipschitz property, Eliasson–Kuksin[13] developed a modifie KAM method to construct quasi-periodic solutions for a more interesting higher dimensional Schrödinger equation. They require a subtle analysis and the introduction of the concept of "Töplitz–Lipschitz matrices" in order to extract asymptotic information on the eigenvalues, which is important to verify the second Melnikov non-resonance conditions. An essential ingredient in Ref. 13 is that finitel many Lipschitz domains cover a neighborhood of ∞. By developing the ideas of Ref. 13 and constructing suitable partial Birkhoff normal form, quasi–periodic solutions of two dimensional cubic Schrödinger equation with periodic boundary conditions are obtained by Geng–Xu–You.[15] By carefully choosing tangential sites $\{i_1,\ldots,i_b\}\in\mathbb{Z}^2$, there is at most one case of complete resonance for $n\in\mathbb{Z}^2\backslash S$, which provides convenience for solving homology equations and obtains no more than fourth-order linear equations, the authors proved that the above nonlinear cubic Schrödinger equation admits a family of small-amplitude quasi-periodic solutions. Procesi–Procesi[30] studied more difficul completely resonant nonlinear Schrödinger equations (NLSs) on $\mathbb{T}^2$ and $\mathbb{T}^d$, respectively. The proof of these above conditions is rather complex and takes fine combinatorial analysis.

Recently, Eliasson–Grébert–Kuksin[35] considered nonlinear beam equation without conservation of momentum

$$u_{tt}+\triangle^2u+mu+\partial_uG(x,u)=0,\quad t\in\mathbb{R},\quad x\in\mathbb{T}^d.$$

They proved the existence of invariant tori for typical m by skillfully choosing admissible set of tangential sites and if $d\geq 2$, then not all the persisted tori are linearly stable. Bernier–Feola–Grébert–Iandoli[36] obtained long-time existence for semi-linear beam equations on irrational Tori. Ge–Geng–Lou[37] proved the existence of KAM tori for a class of two dimensional (2D) non-Hamiltonian completely resonant beam equations with derivative nonlinearities. But there is few result for quasi-periodic solutions of coupled beam system. Shi–Xu[38] proved the existence of a Whitney smooth family of small amplitude quasi-periodic solutions corresponding to finit dimensional invariant tori of an associated infinit dimensional dynamic system for higher dimensional beam equation system

$$\begin{cases} u_{1tt}+\Delta^2u_1+\sigma u_1+u_1u_2^2=0\\ u_{2tt}+\Delta^2u_2+\mu u_2+u_1^2u_2=0\end{cases}$$

under periodic boundary conditions, where $\sigma,\ \mu$ are real external parameters used to avoid resonances. When the equation system have no external parameters, one must deal with the resonances. Bourgain proposed the idea of choosing an appropriate set of tangential sites wisely, such that the Birkhoff normal form Hamiltonian admits quasi-periodic solutions which excite only the Fourier indexes of the tangential sites.

In this paper, we consider the nonlinear completely resonant beam equation system:

$$\begin{cases} u_{1tt}+\Delta^2u_1+u_1|\nabla u_1|^2+u_1^2\Delta u_1+3u_1^2u_2^3=0\\ u_{2tt}+\Delta^2u_2+u_2|\nabla u_2|^2+u_2^2\Delta u_2+3u_1^3u_2^2=0\end{cases},\quad t\in\mathbb{R},x\in\mathbb{T}^2 \tag{1.1}$$

under periodic boundary conditions. We shall use the admissible set of Ref. 37 on

$$\mathbb{Z}^2_{odd}:=\{(n_1,n_2):n_1\in 2\mathbb{Z}-1,n_2\in 2\mathbb{Z}\}$$

which satisfie

Proposition 1 (Structure of admissible set $\mathcal{I}\subset\mathbb{Z}^2_{odd}$).

1. *Any three different elements $i,j,k\in\mathcal{I}$ are not vertices of a rectangle.*
2. *For any $n\in\mathbb{Z}^2_{odd}\backslash\mathcal{I}$, there exists at most one triple $\{i,j,m\}$ with $i,j\in\mathcal{I},m\in\mathbb{Z}^2_{odd}\backslash\mathcal{I}$ such that*

$$\begin{cases} n-m+i-j=0,\\ |n|^2-|m|^2+|i|^2-|j|^2=0.\end{cases}$$

If it exists, we say that n,m are resonant of the firs type. By definition n,m are mutually uniquely determined. We say that (n,m) is a resonant pair of the firs type. In symbols, $n,m\in\mathcal{L}_1$.

3. *For any $n\in\mathbb{Z}^2_{odd}\backslash\mathcal{I}$, there exists at most one triple $\{i,j,m\}$ with $i,j\in\mathcal{I},m\in\mathbb{Z}^2_{odd}\backslash\mathcal{I}$ such that*

$$\begin{cases} n+m-i-j=0,\\ |n|^2+|m|^2-|i|^2-|j|^2=0.\end{cases}$$

If it exists, we say that n,m are resonant of the second type. By definition n,m are mutually uniquely determined. We say that (n,m) is a resonant pair of the second type. In symbols, $n,m\in\mathcal{L}_2$.

4. *Any $n \in \mathbb{Z}^2_{odd}\backslash\mathcal{I}$ is not resonant of both the firs type and the second type, i.e., there exist no $i, j, k, l \in \mathcal{I}$ and $m, m' \in \mathbb{Z}^2_{odd}\backslash\mathcal{I}$, such that*

$$\begin{cases} n - m + i - j = 0, \\ |n|^2 - |m|^2 + |i|^2 - |j|^2 = 0, \\ n + m' - k - l = 0, \\ |n|^2 + |m'|^2 - |k|^2 - |l|^2 = 0. \end{cases}$$

It means that $\mathcal{L}_1 \bigcap \mathcal{L}_2 = \emptyset$.

Assume parameter set $\mathcal{O} \subset \mathbb{R}^{2b}$ is a bounded set with positive Lebesgue measure, and any parameter

$$\xi = (\xi_1, \xi_2) = (\xi_{1i_1}, \ldots, \xi_{1i_b}, \xi_{2i_1}, \ldots, \xi_{2i_b}) \in \mathcal{O}$$

satisfie

$$|\xi_1 - \xi_2|_1 > C.$$

We fin that the normal frequencies of (1.1) have the form

$$\Omega_{hn} = |n|^2 + \frac{1}{2\pi^2}\sum_{i\in\mathcal{I}}\left(\frac{1}{|i|^2} + \frac{1}{|n|^2}\right)\xi_{hi}, \quad n \in \mathbb{Z}^2_{odd}\backslash\mathcal{I}, \quad h = 1, 2.$$

and satisfy a gap condition

$$|\Omega_{1n} - \Omega_{2n}| = \left|\frac{1}{2\pi^2}\sum_{i\in\mathcal{I}}\left(\frac{1}{|i|^2} + \frac{1}{|n|^2}\right)(\xi_{1i} - \xi_{2i})\right| > \gamma,$$

which can help us eliminate the resonance caused by the coupling of equations. The derivative nonlinearity plays an important role in our proof. For nonlinear beam equation without derivative nonlinearity, the normal frequency has the form

$$\Omega_n \sim |n|^2 + \frac{\xi}{n^2}.$$

When n is large enough, the gap condition does not hold. Using this gap condition, coupled system of wave equations with derivative nonlinearity (DNLW) in Refs. 39 and 40 can also be studied by KAM iteration. For example, DNLW in,[39]

$$u_{tt} - u_{xx} + mu + f(\sqrt{-\partial_{xx} + m}\ u) = 0$$

has normal frequency

$$\Omega_n \sim n + \frac{m}{2n} + O(\xi).$$

For DNLW system

$$\begin{cases} u_{tt} - u_{xx} + mu + f(\sqrt{-\partial_{xx} + m}\ u) + P_u(u, v) = 0, \\ v_{tt} - v_{xx} + mv + f(\sqrt{-\partial_{xx} + m}\ v) + P_v(u, v) = 0, \end{cases}$$

where $P(u, v)$ is small perturbation. The normal frequencies satisfy

$$\Omega_{hn} \sim n + \frac{m}{2n} + O(\xi_h), \quad h = 1, 2$$

and

$$|\Omega_{1n} - \Omega_{2n}| \sim O(|\xi_1 - \xi_2|_1).$$

The KAM theorem of Berti–Biasco–Procesi in Ref. 39 should also be valid for DNLW system.

For instance in the more complicate case of quasi-linear coupled wave and beam equation systems in,[41,42]

$$\begin{cases} u_{tt} + u_{xxxx} + P_1(x, t, u_t, u_x, u_{xx}, v_t, v_x, v_{xx}) = 0, \\ v_{tt} - v_{xx} + P_2(x, t, u_t, u_x, u_{xx}, v_t, v_x, v_{xx}) = 0, \end{cases}$$

the KAM method described in this paper does not work.

Then we state our main result.

Theorem 1. *Let $\mathcal{I} = \{i_1, \ldots, i_b\} \subset \mathbb{Z}^2_{odd}$ be an admissible set. There exists a Cantor set $\mathcal{O}$ of positive–measure such that for any $\xi = (\xi_{1i_1}, \ldots, \xi_{2i_b}) \in \mathcal{O}$, the nonlinear completely resonant beam equation system (1.1) admits a small–amplitude, quasi–periodic solution of the form*

$$\begin{cases} u_1(t,x) = \sum_{j=1}^{b} \sqrt{\xi_{1i_j}} \cos(\omega_{1i_j} t + \langle i_j, x\rangle) + O\Big(|\xi_1|^{\frac{3}{2}}\Big) \\ u_2(t,x) = \sum_{j=1}^{b} \sqrt{\xi_{2i_j}} \cos(\omega_{2i_j} t + \langle i_j, x\rangle) + O\Big(|\xi_2|^{\frac{3}{2}}\Big) \end{cases} \tag{1.2}$$

where $\omega_{1i_j} = \varepsilon^{-4}|i_j|^2 + O(|\xi_1|), \omega_{2i_j} = \varepsilon^{-4}|i_j|^2 + O(|\xi_2|)$.

This paper is organized as follows: in Sec. II we give an infinit dimensional KAM theorem; in Sec. III, we give its application to two-dimensional nonlinear completely resonant beam equation system.

II. KAM THEOREM

In this section, we will formulate an infinit dimensional KAM theorem that can be applied to two-dimensional nonlinear completely resonant beam equation system under periodic boundary conditions.

We start by introducing some notation. For given b vectors in $\mathbb{Z}^2_{odd}$, say $\mathcal{I} = \{i_1, \ldots, i_b\}$, we denote $\mathbb{Z}^2_{odd,1} = \mathbb{Z}^2_{odd} \backslash \mathcal{I}$. Let $z_h = (\ldots, z_{hn}, \ldots)_{n \in \mathbb{Z}^2_{odd,1}}$, $h = 1, 2$, and its complex conjugate $\bar{z}_h = (\ldots, \bar{z}_{hn}, \ldots)_{n \in \mathbb{Z}^2_{odd,1}}$. We introduce the weighted norm

$$\|z_h\|_\rho = \sum_{n \in \mathbb{Z}^2_{odd,1}} |z_{hn}| e^{|n|\rho},$$

where $|n| = \sqrt{n_1^2 + n_2^2}$, $n = (n_1, n_2) \in \mathbb{Z}^2_{odd}$ and $\rho > 0$.

Denote a neighborhood of

$$\mathbb{T}^b \times \{I_1 = 0\} \times \{z_1 = 0\} \times \{\bar{z}_1 = 0\} \times \mathbb{T}^b \times \{I_2 = 0\} \times \{z_2 = 0\} \times \{\bar{z}_2 = 0\}$$

by $D_\rho(r,s) :=$

$$\{(\theta_1, I_1, z_1, \bar{z}_1, \theta_2, I_2, z_2, \bar{z}_2) : |\mathrm{Im}\, \theta_h| < r, |I_h| < s^2, \|z_h\|_\rho < s, \|\bar{z}_h\|_\rho < s, h = 1, 2\}, \tag{2.1}$$

where $|\cdot|$ denotes the sup-norm of complex vectors. Moreover, we denote by $\mathcal{O}$ a positive–measure parameter set in $\mathbb{R}^{2b}$.

A function $f : D_\rho(r,s) \times \mathcal{O} \to \mathbb{C}$ is real analytic and C^4_W (i.e., C^4-smooth in the sense of Whitney) in $\xi \in \mathcal{O}$ and has Taylor–Fourier series expansion

$$f(\theta, I, z, \bar{z}; \xi) = \sum_{k \in \mathbb{Z}^{2b}, l \in \mathbb{N}^{2b}, \alpha, \beta \in \mathbb{N}^{2\mathbb{Z}^2_{odd,1}}} f_{kl\alpha\beta}(\xi) e^{\mathrm{i}\langle k, \theta\rangle} I^l z^\alpha \bar{z}^\beta,$$

where $\langle k, \theta\rangle = \sum_{h=1}^{2} \sum_{i \in \mathcal{I}} k_{hi}\theta_{hi}$, $I^l = \prod_{h=1}^{2} \prod_{i \in \mathcal{I}} I_{hi}^{l_{hi}}$ and

$$z^\alpha \bar{z}^\beta = \prod_{j \in \mathbb{Z}^2_{odd,1}} z_{1j}^{\alpha_{1j}} \bar{z}_{1j}^{\beta_{1j}} z_{2j}^{\alpha_{2j}} \bar{z}_{2j}^{\beta_{2j}}, \tag{2.2}$$

$\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{N}^{2\mathbb{Z}^2_{odd,1}}$ have only finitel many nonzero components. We defin the weighted norm of f as follows

$$\|f\|_{D_\rho(r,s), \mathcal{O}} = \sup_{\substack{\|z\|_\rho < s, \|\bar{z}\|_\rho < s \\ h=1,2}} \sum_{k,l,\alpha,\beta} |f_{kl\alpha\beta}|_{\mathcal{O}} e^{|k|r} s^{2|l|} |z^\alpha| |\bar{z}^\beta|, \tag{2.3}$$

where $|f_{kl\alpha\beta}|_{\mathcal{O}} = \sup_{\xi \in \mathcal{O}} \sum_{0 \le i \le 4} |\partial^i_\xi f_{kl\alpha\beta}|$.

To a function F, we associate a Hamiltonian vector fiel define by

$$X_F = \Big(F_I, -F_\theta, \{\mathrm{i}F_{z_n}\}_{n \in \mathbb{Z}^2_{odd,1}}, \{-\mathrm{i}F_{\bar{z}_n}\}_{n \in \mathbb{Z}^2_{odd,1}}\Big).$$

Its weighted norm is define by [the norm $\|\cdot\|_{D_\rho(r,s),\mathcal{O}}$ for scalar functions is define in (2.3). The vector function $G: D_\rho(r,s)\times\mathcal{O}\to\mathbb{C}^m$, $(m<\infty)$ is similarly define as $\|G\|_{D_\rho(r,s),\mathcal{O}}=\sum_{i=1}^m\|G_i\|_{D_\rho(r,s),\mathcal{O}}$]

$$\|X_F\|_{D_\rho(r,s),\mathcal{O}}\equiv\sum_{h=1}^{2}\|F_{I_h}\|_{D_\rho(r,s),\mathcal{O}}+\frac{1}{s^2}\|F_{\theta_h}\|_{D_\rho(r,s),\mathcal{O}}$$
$$+\sup_{D_\rho(r,s)}\Bigg[\frac{1}{s}\sum_{n\in\mathbb{Z}^2_{odd,1}}\|F_{z_{1n}}\|_{\mathcal{O}}e^{|n|\rho}+\frac{1}{s}\sum_{n\in\mathbb{Z}^2_{odd,1}}\|F_{\bar z_{1n}}\|_{\mathcal{O}}e^{|n|\rho} \tag{2.4}$$

$$+\frac{1}{s}\sum_{n\in\mathbb{Z}^2_{odd,1}}\|F_{z_{2n}}\|_{\mathcal{O}}e^{|n|\rho}+\frac{1}{s}\sum_{n\in\mathbb{Z}^2_{odd,1}}\|F_{\bar z_{2n}}\|_{\mathcal{O}}e^{|n|\rho}\Bigg]. \tag{2.5}$$

We now describe a family of Hamiltonians studied in this paper. Let

$$H_0=N+\mathcal{A}+\mathcal{B}+\bar{\mathcal{B}},$$

$$N=\langle\omega(\xi),I\rangle+\sum_{n\in\mathbb{Z}^2_{odd,1}}[\Omega_{1n}(\xi)z_{1n}\bar z_{1n}+\Omega_{2n}(\xi)z_{2n}\bar z_{2n}],$$

$$\mathcal{A}=\sum_{n\in\mathcal{L}_1}Q_n\Big[\sqrt{\xi_{1i}\xi_{1j}}z_{1n}\bar z_{1m}e^{i\theta_{1i}-i\theta_{1j}}+\sqrt{\xi_{2i}\xi_{2j}}z_{2n}\bar z_{2m}e^{i\theta_{2i}-i\theta_{2j}}\Big],$$
$$\mathcal{B}=\sum_{n\in\mathcal{L}_2}\tilde Q_n\Big[\sqrt{\xi_{1i}\xi_{1j}}z_{1n}z_{1m}e^{-i\theta_{1i}-i\theta_{1j}}+\sqrt{\xi_{2i}\xi_{2j}}z_{2n}z_{2m}e^{-i\theta_{2i}-i\theta_{2j}}\Big],$$
$$\overline{\mathcal{B}}=\sum_{n\in\mathcal{L}_2}\tilde Q_n\Big[\sqrt{\xi_{1i}\xi_{1j}}\bar z_{1n}\bar z_{1m}e^{i\theta_{1i}+i\theta_{1j}}+\sqrt{\xi_{2i}\xi_{2j}}\bar z_{2n}\bar z_{2m}e^{i\theta_{2i}+i\theta_{2j}}\Big],$$

where $\xi\in\mathcal{O}$ is a parameter.

The system admits special solutions

$$(\theta_1,0,0,0,\theta_2,0,0,0)\to(\theta_1+\omega_1t,0,0,0,\theta_2+\omega_2t,0,0,0)$$

that corresponds to an invariant torus in the phase space. Consider now the perturbed Hamiltonian

$$H=H_0+P=N+\mathcal{A}+\mathcal{B}+\overline{\mathcal{B}}+P(\theta,I,z,\bar z,\xi).$$

Our goal is to prove that, for most values of parameter $\xi=(\xi_1,\xi_2)\in\mathcal{O}$ (in Lebesgue measure sense), the Hamiltonians $H=N+\mathcal{A}+\mathcal{B}+\overline{\mathcal{B}}+P$ still admit invariant tori provided that $\|X_P\|_{D_\rho(r,s),\mathcal{O}}$ is sufficientl small. One should not expect a KAM theorem for general infinit dimensional Hamiltonian systems.

For this purpose, we need the following six assumptions:

(A1) *Nondegeneracy*: The map $\xi\mapsto\omega(\xi)$ is a C^4_W diffeomorphism between $\mathcal{O}$ and its image.

(A2) *Asymptotics of normal frequencies*:

$$\Omega_{hn}=\varepsilon^{-4}(|n|^2)+\Omega^0_{hn},\ n\in\mathbb{Z}^2_{odd,1},\quad h=1,2, \tag{2.6}$$

where $\Omega^0_{hn}\in C^4_W(\mathcal{O})$ with C^4_W-norm bounded by a positive constant L.

(A2*) *Gap condition of normal frequencies*: There exist $\gamma>0$, such that

$$|\Omega^0_{1n}-\Omega^0_{2n}|>\gamma,\quad n\in\mathbb{Z}^2_{odd,1}. \tag{2.7}$$

We fin that the normal frequencies of Eq. (2.7) satisfy a gap condition which can help us eliminate the resonance caused by the coupling of equations.

(A3) *Melnikov's non–resonance conditions*: Denote

$$M_{hn}=\Omega_{hn},\ n\notin\mathbb{Z}_{odd,1}\backslash(\mathcal{L}_1\cup\mathcal{L}_2),$$

$$M_{hn}=\begin{pmatrix}\Omega_{hn}+\omega_{hi} & Q_n\\ Q_m & \Omega_{hm}+\omega_{hj}\end{pmatrix}\ n\in\mathcal{L}_1,$$

$$M_{hn} = \begin{pmatrix} \Omega_{hn} - \omega_{hi} & -\tilde{Q}_n \\ \tilde{Q}_m & -(\Omega_{hm} - \omega_{hj}) \end{pmatrix} \ n \in \mathcal{L}_2.$$

Suppose $Q_n, \tilde{Q}_n, Q_m, \tilde{Q}_m \in C^4_W(\mathcal{O})$ and there exist $\tau > 0$, such that

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{|k|^\tau}, \ k \neq 0, \tag{2.8}$$

$$|\det(\langle k, \omega \rangle I_{\varphi(n)} \pm M_{hn})| \geq \frac{\gamma}{|k|^\tau}, \tag{2.9}$$

$$|\det(\langle k, \omega \rangle I_{\varphi(n)\varphi(n')} \pm M_{hn} \otimes I_{\varphi(n')} \pm I_{\varphi(n)} \otimes M_{h'n'})| \geq \frac{\gamma}{|k|^\tau}, \ k \neq 0, \tag{2.10}$$

where I_a is $a \times a$ identity matrix, $\varphi(n)(resp.\varphi(n'))$ denotes the dimension of $M_{hn}(resp.M_{hn'})$. $\det(\cdot)$ and $\otimes$ denotes the determinant and the tensor product respectively.

(A4) *Regularity of* $\mathcal{A} + \mathcal{B} + \bar{\mathcal{B}} + P$: $\mathcal{A} + \mathcal{B} + \bar{\mathcal{B}} + P$ is real analytic in $I, \theta, z, \bar{z}$ and C^4_W-smooth in ξ. In addition

$$\|X_{\mathcal{A}}\|_{D(r,s),\mathcal{O}} + \|X_{\mathcal{B}}\|_{D(r,s),\mathcal{O}} + \|X_{\bar{\mathcal{B}}}\|_{D(r,s),\mathcal{O}} < 1, \|X_P\|_{s;D(r,s),\mathcal{O}} < \varepsilon.$$

(A5) *Momentum conservation property of the perturbation*: $\mathcal{A} + \mathcal{B} + \overline{\mathcal{B}} + P$ admits momentum conservation such that

$$\mathcal{A} + \mathcal{B} + \overline{\mathcal{B}} + P = \sum_{k\in\mathbb{Z}^{2b}, l\in\mathbb{N}^{2b}, \alpha,\beta\in\mathbb{N}^{2\mathbb{Z}^2_{odd,1}}} (\mathcal{A} + \mathcal{B} + \overline{\mathcal{B}} + P)_{kl\alpha\beta}(\xi) I^l e^{\mathrm{i}\langle k,\theta\rangle} z^\alpha \bar{z}^\beta,$$

where k, α, β have the following relation

$$\sum_{h=1}^{2} \left(\sum_{j=1}^{b} k_{hj} i_j + \sum_{n\in\mathbb{Z}^2_{odd,1}} (\alpha_{hn} - \beta_{hn}) n \right) = 0.$$

(A6) *Töplitz–Lipschitz property*: For any fixe $n, m \in \mathbb{Z}^2_{odd}, c \in \mathbb{Z}^2_{odd} \backslash \{0\}$, $h = 1, 2$, the limits

$$\lim_{t\to\infty} \frac{\partial^2(\mathcal{B} + P)}{\partial z_{hn+tc} \partial z_{hm-tc}}, \quad \lim_{t\to\infty} \frac{\partial^2 \left(\sum_{n\in\mathbb{Z}^2_{odd,1}} \Omega^0_{hn} z_{hn} \bar{z}_{hn} + \mathcal{A} + P \right)}{\partial z_{hn+tc} \partial \bar{z}_{hm+tc}}, \quad \lim_{t\to\infty} \frac{\partial^2(\overline{\mathcal{B}} + P)}{\partial \bar{z}_{hn+tc} \partial \bar{z}_{hm-tc}}$$

exist. Moreover, there exists $K > 0$, such that when $t > K, N + \mathcal{A} + \mathcal{B} + \overline{\mathcal{B}} + P$ satisfie

$$\left\| \frac{\partial^2 \mathcal{H}_h}{\partial z_{hn+tc} \partial \bar{z}_{hm+tc}} - \lim_{t\to\infty} \frac{\partial^2 \mathcal{H}_h}{\partial z_{hn+tc} \partial \bar{z}_{hm+tc}} \right\|_{D_\rho(r,s),\mathcal{O}} \leq \frac{\varepsilon}{t} e^{-|n-m|\rho},$$

with

$$\mathcal{H}_h := \sum_{n\in\mathbb{Z}^2_{odd,1}} \Omega^0_{hn} z_{hn} \bar{z}_{hn} + \mathcal{A} + P,$$

$$\left\| \frac{\partial^2(\mathcal{B} + P)}{\partial z_{hn+tc} \partial z_{hm-tc}} - \lim_{t\to\infty} \frac{\partial^2(\mathcal{B} + P)}{\partial z_{hn+tc} \partial z_{hm-tc}} \right\|_{D_{\rho(r,s)},\mathcal{O}} \leq \frac{\varepsilon}{t} e^{-|n+m|\rho},$$

$$\left\| \frac{\partial^2(\overline{\mathcal{B}} + P)}{\partial \bar{z}_{hn+tc} \partial \bar{z}_{hm-tc}} - \lim_{t\to\infty} \frac{\partial^2(\overline{\mathcal{B}} + P)}{\partial \bar{z}_{hn+tc} \partial \bar{z}_{hm-tc}} \right\|_{D_{\rho(r,s)},\mathcal{O}} \leq \frac{\varepsilon}{t} e^{-|n+m|\rho}.$$

Now we are ready to state an infinit dimensional KAM Theorem.

Theorem 2. *Assume that the Hamiltonian $H_0 + P$ satisfie (A1)–(A6). Let $\gamma > 0$ be small enough. Then there is a positive constant ε, depending on $b, L, K, \tau, \gamma, r, s$ and ρ such that if $\|X_P\|_{D_\rho(r,s),\mathcal{O}} < \varepsilon$, the following holds: There exist a Cantor subset $\mathcal{O}_\gamma \subset \mathcal{O}$ with meas $(\mathcal{O} \backslash \mathcal{O}_\gamma) = O\left(\gamma^{\frac{1}{4}}\right)$ and two maps (analytic in θ and C^4_W in ξ)*

$$\Psi : \mathbb{T}^{2b} \times \mathcal{O}_\gamma \to D_\rho(r,s), \quad \tilde{\omega} : \mathcal{O}_\gamma \to \mathbb{R}^{2b},$$

where Ψ is $\frac{\varepsilon}{\gamma^4}$-close to the trivial embedding $\Psi_0 : \mathbb{T}^{2b} \times \mathcal{O} \to \mathbb{T}^{2b} \times \{0,0,0,0,0,0\}$ and $\tilde{\omega}$ is ε-close to the unperturbed frequency ω, such that for any $\xi \in \mathcal{O}_\gamma$ and $\theta \in \mathbb{T}^{2b}$, the curve $t \to \Psi(\theta + \tilde{\omega}(\xi)t, \xi)$ is a quasi-periodic solution of the equations governed by $H_0 + P$.

Because $(A2^*)$ conditions lead to the cases $k = 0, h \neq h', n = n'$ in (2.10), the Proof of Theorem 2 is almost identical to the proof in Ref. 15. Thus we do not repeat the KAM steps in this paper.

III. APPLICATION TO THE COUPLED BEAM EQUATION SYSTEM

We consider the two–dimensional nonlinear completely resonant beam equation system

$$\begin{cases} u_{1tt} + \Delta^2 u_1 + u_1|\nabla u_1|^2 + u_1^2 \Delta u_1 + 3u_1^2 u_2^3 = 0 \\ u_{2tt} + \Delta^2 u_2 + u_2|\nabla u_2|^2 + u_2^2 \Delta u_2 + 3u_1^3 u_2^2 = 0 \end{cases}, \quad t \in \mathbb{R}, x \in \mathbb{T}^2 \tag{3.1}$$

with periodic boundary conditions

$$u_h(t, x_1 + 2\pi, x_2) = u_h(t, x_1, x_2 + 2\pi) = u_h(t, x_1, x_2), \quad h = 1, 2.$$

Scaling $u_h \to \varepsilon^{1/2} u_h, h = 1, 2$. Let $v_h = u_{ht}$ and

$$w_h = \frac{(-\Delta)^{\frac{1}{2}}}{\sqrt{2}} u_h - \frac{\mathrm{i}(-\Delta)^{-\frac{1}{2}}}{\sqrt{2}} v_h.$$

For $w = (w_1, w_2)$, we have

$$w_t = \mathrm{i}\frac{\partial H}{\partial \bar{w}}$$

and the corresponding Hamiltonian function is

$$H = H_1 + H_2 + \varepsilon^2 \tilde{G},$$

where

$$H_1 = \int_{\mathbb{T}^2} [(-\Delta) w_1] \bar{w}_1 dx + \frac{\varepsilon}{2} \int_{\mathbb{T}^2} \left[(-\Delta)^{-\frac{1}{2}} \left(\frac{w_1 + \bar{w}_1}{\sqrt{2}}\right)\right]^2 \left[(-\Delta)^{-\frac{1}{2}} \left(\frac{\nabla w_1 + \nabla \bar{w}_1}{\sqrt{2}}\right)\right]^2 dx,$$
$$H_2 = \int_{\mathbb{T}^2} [(-\Delta) w_2] \bar{w}_2 dx + \frac{\varepsilon}{2} \int_{\mathbb{T}^2} \left[(-\Delta)^{-\frac{1}{2}} \left(\frac{w_2 + \bar{w}_2}{\sqrt{2}}\right)\right]^2 \left[(-\Delta)^{-\frac{1}{2}} \left(\frac{\nabla w_2 + \nabla \bar{w}_2}{\sqrt{2}}\right)\right]^2 dx,$$
$$\tilde{G} = \frac{1}{3} \int_{\mathbb{T}^2} \left[(-\Delta)^{-\frac{1}{2}} \left(\frac{w_1 + \bar{w}_1}{\sqrt{2}}\right)\right]^3 \left[(-\Delta)^{-\frac{1}{2}} \left(\frac{w_2 + \bar{w}_2}{\sqrt{2}}\right)\right]^3 dx.$$

Let tangential sites $\mathcal{I} = \{i_1, \ldots, i_b\} \subset \mathbb{Z}^2_{odd}$. Under periodic boundary conditions, we denote the eigenvalues of Δ by λ_n, $n \in \mathbb{Z}^2_{odd}$,

$$\omega_{hi_j} = \lambda_{i_j} = |i_j|^2, \quad 1 \le j \le b, \quad \Omega_{hn} = \lambda_n = |n|^2, \quad n \in \mathbb{Z}^2_{odd,1},$$

and the corresponding eigenfunctions $\phi_n(x) = \frac{1}{2\pi} e^{\mathrm{i}\langle n, x\rangle}$.

Expending

$$w_h(x) = \sum_{n \in \mathbb{Z}^2_{odd}} q_{hn} \phi_n(x), \quad \bar{w}(x) = \sum_{n \in \mathbb{Z}^2_{odd}} \bar{q}_{hn} \phi_{-n}(x),$$

system takes the lattice form

$$\dot{q}_{hn} = \mathrm{i}\left(\lambda_n q_{hn} + \varepsilon \frac{\partial G}{\partial \bar{q}_{hn}} + \varepsilon^2 \frac{\partial \tilde{G}}{\partial \bar{q}_{hn}}\right)$$
$$G(q_h, \bar{q}_h) = \frac{1}{2} \int_{\mathbb{T}^2} \left[(-\Delta)^{-\frac{1}{2}} \left(\frac{w_h + \bar{w}_h}{\sqrt{2}}\right)\right]^2 \left[(-\Delta)^{-\frac{1}{2}} \left(\frac{\nabla w_h + \nabla \bar{w}_h}{\sqrt{2}}\right)\right]^2 dx$$

and

$$\tilde{G}(q_1, \bar{q}_1, q_2, \bar{q}_2) = \frac{1}{3} \int_{\mathbb{T}^2} \left[(-\Delta)^{-\frac{1}{2}} \left(\frac{w_1 + \bar{w}_1}{\sqrt{2}}\right)\right]^3 \left[(-\Delta)^{-\frac{1}{2}} \left(\frac{w_2 + \bar{w}_2}{\sqrt{2}}\right)\right]^3 dx$$

with corresponding Hamiltonian function $H = H_1 + H_2 + \varepsilon^2 \tilde{G}$ with

$$H_h = \sum_{n\in\mathbb{Z}^2_{odd}} \lambda_n q_{hn}\bar{q}_{hn} + \varepsilon G(q_{hn}, \bar{q}_{hn}).$$

We have

$$G(q_h, \bar{q}_h) = \sum_{\substack{\alpha,\beta \\ |\alpha|+|\beta|=4}} G_{h\alpha\beta} q_h^{\alpha} \bar{q}_h^{\beta}$$

and

$$\tilde{G}(q_1, \bar{q}_1, q_2, \bar{q}_2) = \sum_{\substack{\alpha,\beta,\tilde{\alpha},\tilde{\beta} \\ |\alpha|+|\beta|=3 \\ |\tilde{\alpha}|+|\tilde{\beta}|=3}} \tilde{G}_{\alpha\beta\tilde{\alpha}\tilde{\beta}} q_1^{\alpha}\bar{q}_1^{\beta} q_2^{\tilde{\alpha}}\bar{q}_2^{\tilde{\beta}}.$$

Then we have $G_{h\alpha\beta} = 0$, if $\sum_{n\in\mathbb{Z}^2_{odd}} (\alpha_n - \beta_n)n \neq 0$ and $\tilde{G}(q_1, \bar{q}_1, q_2, \bar{q}_2) = 0$ if

$$\sum_{n\in\mathbb{Z}^2_{odd}} (\alpha_n - \beta_n)n + \sum_{n\in\mathbb{Z}^2_{odd}} \left(\tilde{\alpha}_n - \tilde{\beta}_n\right)n \neq 0.$$

Like Poschel's conclusion,[27] we ignore the constant factor $\frac{1}{2}$. Thus the function $G = G_1 + G_2$ in the above takes the form:

$$\begin{aligned}
G_h(q, \bar{q}) &= \sum_{i+j+k+l=0} \frac{-\langle k, l\rangle}{4(2\pi)^2\sqrt{\lambda_i\lambda_j\lambda_k\lambda_l}} \left(q_{hi}q_{hj}q_{hk}q_{hl} + \bar{q}_{hi}\bar{q}_{hj}\bar{q}_{hk}\bar{q}_{hl}\right) \\
&+ \sum_{i+j+k-l=0} \frac{\langle k, l\rangle - \langle j, k\rangle}{2(2\pi)^2\sqrt{\lambda_i\lambda_j\lambda_k\lambda_l}} \left(q_{hi}q_{hj}q_{hk}\bar{q}_{hl} + \bar{q}_{hi}\bar{q}_{hj}\bar{q}_{hk}q_{hl}\right) \\
&+ \sum_{i-j-k+l=0} \frac{4\langle k, l\rangle - \langle j, k\rangle - \langle i, l\rangle}{4(2\pi)^2\sqrt{\lambda_i\lambda_j\lambda_k\lambda_l}} \left(q_{hi}\bar{q}_{hj}\bar{q}_{hk}q_{hl} + \bar{q}_{hi}q_{hj}q_{hk}\bar{q}_{hl}\right) \\
&:= \sum_{\substack{\sigma_1 i+\sigma_2 j+\sigma_3 k+\sigma_4 l=0 \\ \sigma_1,\sigma_2,\sigma_3,\sigma_4=\pm}} g_{h,\sigma_1 i,\sigma_2 j,\sigma_3 k,\sigma_4 l} q_{hi}^{\sigma_1} q_{hj}^{\sigma_2} q_{hk}^{\sigma_3} q_{hl}^{\sigma_4}
\end{aligned}$$

By direct computation, one can verify that the gradient of $G, \tilde{G}$ satisfie the following regularity property.

Lemma 3.1. For any fixe $\rho > 0$, $\tilde{G}_{\bar{q}}$ and $G_{\bar{q}}$ are real analytic as a map in a neighborhood of the origin with

$$\|G_{\bar{q}}\|_{\rho} \leq c\|q\|_{\rho}^3$$

and

$$\|\tilde{G}_{\bar{q}}\|_{\rho} \leq c\|q\|_{\rho}^5.$$

For an admissible set of tangential site $\mathcal{I} = \{i_1, \ldots, i_b\} \subset \mathbb{Z}^2_{odd}$, we have a nice normal form for H.

Proposition 2. Let $\mathcal{I}$ be admissible. For Hamiltonian function H, there exists a symplectic transformation Ψ such that

$$H \circ \Psi = \langle \omega(\xi), I\rangle + \sum_{n\in\mathbb{Z}^2_{odd,1}} \left[\Omega_{1n}(\xi)z_{1n}\bar{z}_{1n} + \Omega_{2n}(\xi)z_{2n}\bar{z}_{2n}\right] + \mathcal{A} + \mathcal{B} + \overline{\mathcal{B}} + P,$$

where

$$\begin{aligned}
\omega_{hi} &= \varepsilon^{-4}|i|^2 + \frac{1}{4\pi^2|i|^2}\xi_{hn} + \frac{1}{2\pi^2}\sum_{j\in\mathcal{I}, j\neq i}\left(\frac{1}{|i|^2} + \frac{1}{|j|^2}\right)\xi_{hj}, \quad i \in \mathcal{I}, \\
\Omega_{hn} &= \varepsilon^{-4}|n|^2 + \frac{1}{2\pi^2}\sum_{i\in\mathcal{I}}\left(\frac{1}{|i|^2} + \frac{1}{|n|^2}\right)\xi_{hi}, \quad n \in \mathbb{Z}^2_{odd,1}, \\
\mathcal{A} &= \sum_{n\in\mathcal{L}_1} Q_n\left[\sqrt{\xi_{1i}\xi_{1j}}z_{1n}\bar{z}_{1m}e^{i\theta_{1i} - i\theta_{1j}} + \sqrt{\xi_{2i}\xi_{2j}}z_{2n}\bar{z}_{2m}e^{i\theta_{2i} - i\theta_{2j}}\right], \\
\mathcal{B} &= \sum_{n\in\mathcal{L}_2} \tilde{Q}_n\left[\sqrt{\xi_{1i}\xi_{1j}}z_{1n}z_{1m}e^{-i\theta_{1i} - i\theta_{1j}} + \sqrt{\xi_{2i}\xi_{2j}}z_{2n}z_{2m}e^{-i\theta_{2i} - i\theta_{2j}}\right],
\end{aligned}$$

$$
\begin{aligned}
\overline{\mathcal{B}} &= \sum_{n\in\mathcal{L}_2}\tilde{Q}_n\Big[\sqrt{\xi_{1i}\xi_{1j}}\bar{z}_{1n}\bar{z}_{1m}e^{i\theta_{1i}+i\theta_{1j}}+\sqrt{\xi_{2i}\xi_{2j}}\bar{z}_{2n}\bar{z}_{2m}e^{i\theta_{2i}+i\theta_{2j}}\Big],\\
|P| &= O\Big(\varepsilon^2|I|^2+\varepsilon^2|I|\|z\|_\rho^2+\varepsilon\xi^{\frac{1}{2}}\|z\|_\rho^3+\varepsilon^2\|z\|_\rho^4+\varepsilon^2\xi^3+\varepsilon^3\xi^{\frac{5}{2}}\|z\|_\rho\\
&\quad+\varepsilon^4\xi^2\|z\|_\rho^2+\varepsilon^5\xi^{\frac{3}{2}}\|z\|_\rho^3\Big).
\end{aligned}
$$

and where

$$
\begin{cases}
Q_n = \dfrac{4\langle i,j\rangle+4\langle m,n\rangle+4\langle j,n\rangle+4\langle i,m\rangle-4\langle j,m\rangle-4\langle i,n\rangle}{(2\pi)^2|i||j||m||n|}\\
\tilde{Q}_n = \dfrac{4\langle j,n\rangle-\langle m,n\rangle-\langle i,j\rangle}{(2\pi)^2|i||j||m||n|}.
\end{cases}
$$

Remark 3.1. The term $\tilde{G}$ only provide couple terms in P, thus we do not calculate its normal form.

Proof. The proof consists of several symplectic change of variables. For convenience, we defin three sets as following:

$$
S_1=\left\{(i,j,m):\begin{smallmatrix} i+j+n+m=0\\ |i|^2+|j|^2+|n|^2+|m|^2\neq 0\\ \sharp\mathcal{I}\cap\{i,j,n,m\}\geq 2\end{smallmatrix}\right\},
$$

and similarly

$$
S_2=\left\{(i,j,m):\begin{smallmatrix} i+j-n+m=0\\ |i|^2+|j|^2-|n|^2+|m|^2\neq 0\\ \sharp\mathcal{I}\cap\{i,j,n,m\}\geq 2\end{smallmatrix}\right\},
$$

$$
S_3=\left\{(i,j,m):\begin{smallmatrix} i-j+n-m=0\\ |i|^2-|j|^2+|n|^2-|m|^2\neq 0\\ \sharp\mathcal{I}\cap\{i,j,n,m\}\geq 2\end{smallmatrix}\right\}.
$$

Firstly, let

$$
\begin{aligned}
F &= \sum_{S_1,h}\frac{i\varepsilon}{\lambda_i+\lambda_j+\lambda_n+\lambda_m}g_{h,i,j,n,m}\big(q_{hi}q_{hj}q_{hn}q_{hm}+\bar{q}_{hi}\bar{q}_{hj}\bar{q}_{hn}\bar{q}_{hm}\big)\\
&\quad+\sum_{S_2,h}\frac{i\varepsilon}{\lambda_i+\lambda_j-\lambda_n+\lambda_m}g_{h,i,j,-n,m}\big(q_{hi}q_{hj}\bar{q}_{hn}q_{hm}+\bar{q}_{hi}\bar{q}_{hj}q_{hn}\bar{q}_{hm}\big)\\
&\quad+\sum_{S_3,h}\frac{i\varepsilon}{\lambda_i-\lambda_j+\lambda_n-\lambda_m}g_{h,i,-j,n,-m}\big(q_{hi}\bar{q}_{hj}q_{hn}\bar{q}_{hm}+\bar{q}_{hi}q_{hj}\bar{q}_{hn}q_{hm}\big)
\end{aligned}
\tag{3.2}
$$

and X_F^1 be the time one map of the flo of the associated Hamiltonian systems. The change of variables X_F^1 sends H to

$$
\begin{aligned}
H\circ X_F^1 &= H+\{H,F\}+\int_0^1(1-t)\{\{H,F\},F\}\circ\phi_F^t dt\\
&= \sum_{i\in\mathcal{I}}\lambda_i|q_{1i}|^2+\sum_{n\in\mathbb{Z}^2_{odd,1}}\lambda_n|q_{1n}|^2+\sum_{i\in\mathcal{I}}\frac{\varepsilon}{8\pi^2|i|^2}|q_{1i}|^4\\
&\quad+\sum_{i,j\in\mathcal{I},i\neq j}\frac{\varepsilon}{2\pi^2|i|^2}|q_{1i}|^2|q_{1j}|^2+\sum_{i\in\mathcal{I},j\in\mathbb{Z}^2_{odd,1}}\frac{\varepsilon}{2\pi^2|i|^2}|q_{1i}|^2|q_{1j}|^2\\
&\quad+\sum_{i\in\mathcal{I},j\in\mathbb{Z}^2_{odd,1}}\frac{\varepsilon}{2\pi^2|j|^2}|q_{1i}|^2|q_{1j}|^2\\
&\quad+\sum_{n\in\mathcal{L}_1}\varepsilon Q_n\big(q_{1i}\bar{q}_{1j}q_{1n}\bar{q}_{1m}+\bar{q}_{1i}q_{1j}\bar{q}_{1n}q_{1m}\big)+\sum_{n\in\mathcal{L}_2}\varepsilon\tilde{Q}_n\big(q_{1i}q_{1j}\bar{q}_{1n}\bar{q}_{1m}+\bar{q}_{1i}\bar{q}_{1j}q_{1n}q_{1m}\big)\\
&\quad+\sum_{i\in\mathcal{I}}\lambda_i|q_{2i}|^2+\sum_{n\in\mathbb{Z}^2_{odd,1}}\lambda_n|q_{2n}|^2+\sum_{i\in\mathcal{I}}\frac{\varepsilon}{8\pi^2|i|^2}|q_{2i}|^4\\
&\quad+\sum_{i,j\in\mathcal{I},i\neq j}\frac{\varepsilon}{2\pi^2|i|^2}|q_{2i}|^2|q_{2j}|^2+\sum_{i\in\mathcal{I},j\in\mathbb{Z}^2_{odd,1}}\frac{\varepsilon}{2\pi^2|i|^2}|q_{2i}|^2|q_{2j}|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i\in\mathcal{I},j\in\mathbb{Z}^2_{odd,1}}\frac{\varepsilon}{2\pi^2|j|^2}|q_{2i}|^2|q_{2j}|^2\\
&+\sum_{n\in\mathcal{L}_1}\varepsilon Q_n(q_{2i}\bar{q}_{2j}q_{2n}\bar{q}_{2m}+\bar{q}_{2i}q_{2j}\bar{q}_{2n}q_{2m})+\sum_{n\in\mathcal{L}_2}\varepsilon\tilde{Q}_n(q_{2i}q_{2j}\bar{q}_{2n}\bar{q}_{2m}+\bar{q}_{2i}\bar{q}_{2j}q_{2n}q_{2m})\\
&+O(|q|\|w\|_\rho^3+\|w\|_\rho^4+|q|^6+|q|^5\|w\|_\rho+|q|^4\|w\|_\rho^2+|q|^3\|w\|_\rho^3).
\end{aligned}
$$

where

$$
\begin{cases}
Q_n=\dfrac{4\langle i,j\rangle+4\langle m,n\rangle+4\langle j,n\rangle+4\langle i,m\rangle-4\langle j,m\rangle-4\langle i,n\rangle}{(2\pi)^2|i||j||m||n|}\\
\tilde{Q}_n=\dfrac{4\langle j,n\rangle-\langle m,n\rangle-\langle i,j\rangle}{(2\pi)^2|i||j||m||n|}
\end{cases}.
$$

We remind that (n,m) are resonant pairs and (i,j) is uniquely determined by (n,m). Here we need to state a fact: The set

$$
\left\{(i,j,n,m)\in\mathbb{Z}^2_{odd}:\begin{array}{c}i+j-n+m=0\\|i|^2+|j|^2-|n|^2+|m|^2\neq 0\end{array}\right\}
$$

do not exist in $\mathbb{Z}^2_{odd}$ thanks to the structure of $\mathbb{Z}^2_{odd}$. The next thing to do in the proof is introduce standard action-angle variables in the tangential space

$$
q_{hi}=\sqrt{I_{hi}+\xi_{hi}}e^{i\theta_{hi}},\quad \bar{q}_{hi}=\sqrt{I_{hi}+\xi_{hi}}e^{-i\theta_{hi}},\quad i\in\mathcal{I},
$$

and

$$
q_{hn}=z_{hn},\bar{q}_{hn}=\bar{z}_{hn},n\in\mathbb{Z}^2_{odd,1}.
$$

Then

$$
\begin{aligned}
H\circ X_F^1=&\sum_{i\in\mathcal{I}}\left[\lambda_i+\frac{\varepsilon}{4\pi^2|i|^2}\xi_{1i}+\frac{\varepsilon}{2\pi^2}\sum_{j\in\mathcal{I},j\neq i}\left(\frac{1}{|i|^2}+\frac{1}{|j|^2}\right)\xi_{1j}\right]I_{1i}\\
&+\sum_{n\in\mathbb{Z}^2_{odd,1}}\left[\lambda_n+\frac{\varepsilon}{2\pi^2}\sum_{i\in\mathcal{I}}\left(\frac{1}{|i|^2}+\frac{1}{|n|^2}\right)\xi_{1i}\right]z_{1n}\bar{z}_{1n}\\
&+\sum_{i\in\mathcal{I}}\left[\lambda_i+\frac{\varepsilon}{4\pi^2|i|^2}\xi_{2i}+\frac{\varepsilon}{2\pi^2}\sum_{j\in\mathcal{I},j\neq i}\left(\frac{1}{|i|^2}+\frac{1}{|j|^2}\right)\xi_{2j}\right]I_{2i}\\
&+\sum_{n\in\mathbb{Z}^2_{odd,1}}\left[\lambda_n+\frac{\varepsilon}{2\pi^2}\sum_{i\in\mathcal{I}}\left(\frac{1}{|i|^2}+\frac{1}{|n|^2}\right)\xi_{2i}\right]z_{2n}\bar{z}_{2n}\\
&+\sum_{n\in\mathcal{L}_1}\varepsilon Q_n\left[\sqrt{\xi_{1i}\xi_{1j}}z_{1n}\bar{z}_{1m}e^{i\theta_{1i}-i\theta_{1j}}+\sqrt{\xi_{2i}\xi_{2j}}z_{2n}\bar{z}_{2m}e^{i\theta_{2i}-i\theta_{2j}}\right]\\
&+\sum_{n\in\mathcal{L}_2}\varepsilon\tilde{Q}_n\left[\sqrt{\xi_{1i}\xi_{1j}}(z_{1n}z_{1m}e^{-i\theta_{1i}-i\theta_{1j}}+\bar{z}_{1n}\bar{z}_{1m}e^{i\theta_{1i}+i\theta_{1j}})\right.\\
&\left.+\sqrt{\xi_{2i}\xi_{2j}}(z_{2n}z_{2m}e^{-i\theta_{2i}-i\theta_{2j}}+\bar{z}_{2n}\bar{z}_{2m}e^{i\theta_{2i}+i\theta_{2j}})\right]\\
&+O\left(\varepsilon|I|^2+\varepsilon|I|\|z\|_\rho^2+\varepsilon\xi^{\frac{1}{2}}\|z\|_\rho^3+\varepsilon\|z\|_\rho^4+\varepsilon^2\xi^3+\varepsilon^2\xi^{\frac{5}{2}}\|z\|_\rho\right.\\
&\left.+\varepsilon^2\xi^2\|z\|_\rho^2+\varepsilon^2\xi^{\frac{3}{2}}\|z\|_\rho^3\right).
\end{aligned}
$$

Finally, by the scaling in time

$$
\xi\to\varepsilon^3\xi,\quad I\to\varepsilon^5 I,\quad \theta\to\theta,\quad z\to\varepsilon^{\frac{5}{2}}z,\quad \bar{z}\to\varepsilon^{\frac{5}{2}}\bar{z},
$$

we finall arrive at the rescaled Hamiltonian

$$
\begin{aligned}
H&=\varepsilon^{-9}H\left(\varepsilon^3\xi,\varepsilon^5 I,\theta,\varepsilon^{\frac{5}{2}}z,\varepsilon^{\frac{5}{2}}\bar{z}\right)\\
&=\langle\omega(\xi),I\rangle+\sum_{n\in\mathbb{Z}^2_{odd,1}}[\Omega_{1n}(\xi)z_{1n}\bar{z}_{1n}+\Omega_{2n}(\xi)z_{2n}\bar{z}_{2n}]+\mathcal{A}+\mathcal{B}+\overline{\mathcal{B}}+P,
\end{aligned}
$$

where

$$\omega_{hi} = \varepsilon^{-4}|i|^2 + \frac{1}{4\pi^2|i|^2}\xi_{hi} + \frac{1}{2\pi^2}\sum_{j\in\mathcal{I},j\neq i}\left(\frac{1}{|i|^2}+\frac{1}{|j|^2}\right)\xi_{hj}, \quad i\in\mathcal{I},$$

$$\Omega_{hn} = \varepsilon^{-4}|n|^2 + \frac{1}{2\pi^2}\sum_{i\in\mathcal{I}}\left(\frac{1}{|i|^2}+\frac{1}{|n|^2}\right)\xi_{hi}, \quad n\in\mathbb{Z}^2_{odd,1},$$

$$\mathcal{A} = \sum_{n\in\mathcal{L}_1} Q_n\Big[\sqrt{\xi_{1i}\xi_{1j}}z_{1n}\bar{z}_{1m}e^{i\theta_{1i}-i\theta_{1j}} + \sqrt{\xi_{2i}\xi_{2j}}z_{2n}\bar{z}_{2m}e^{i\theta_{2i}-i\theta_{2j}}\Big],$$

$$\mathcal{B} = \sum_{n\in\mathcal{L}_2} \tilde{Q}_n\Big[\sqrt{\xi_{1i}\xi_{1j}}z_{1n}z_{1m}e^{-i\theta_{1i}-i\theta_{1j}} + \sqrt{\xi_{2i}\xi_{2j}}z_{2n}z_{2m}e^{-i\theta_{2i}-i\theta_{2j}}\Big],$$

$$\overline{\mathcal{B}} = \sum_{n\in\mathcal{L}_2} \tilde{Q}_n\Big[\sqrt{\xi_{1i}\xi_{1j}}\bar{z}_{1n}\bar{z}_{1m}e^{i\theta_{1i}+i\theta_{1j}} + \sqrt{\xi_{2i}\xi_{2j}}\bar{z}_{2n}\bar{z}_{2m}e^{i\theta_{2i}+i\theta_{2j}}\Big],$$

$$|P| = O\Big(\varepsilon^2|I|^2 + \varepsilon^2|I|\|z\|_\rho^2 + \varepsilon\xi^{\frac{1}{2}}\|z\|_\rho^3 + \varepsilon^2\|z\|_\rho^4 + \varepsilon^2\xi^3 + \varepsilon^3\xi^{\frac{5}{2}}\|z\|_\rho$$
$$+ \varepsilon^4\xi^2\|z\|_\rho^2 + \varepsilon^5\xi^{\frac{3}{2}}\|z\|_\rho^3\Big).$$

Next let us verify that $H = N + \mathcal{A} + \mathcal{B} + \bar{\mathcal{B}} + P$ satisfie the assumptions $(A1)$–$(A6)$.

Verificatio of $(A1)$:

$$\frac{\partial\omega}{\partial\xi} = \begin{pmatrix} \frac{\partial\omega_1}{\partial\xi_1} & \frac{\partial\omega_1}{\partial\xi_2} \\ \frac{\partial\omega_2}{\partial\xi_1} & \frac{\partial\omega_2}{\partial\xi_2}\end{pmatrix}$$

$$\frac{\partial\omega_1}{\partial\xi_1} = \frac{\partial\omega_2}{\partial\xi_2} = \frac{1}{4\pi^2}\begin{pmatrix} \frac{1}{|i_1|^2} & \frac{2}{|i_1|^2}+\frac{2}{|i_2|^2} & \cdots & \frac{2}{|i_1|^2}+\frac{2}{|i_b|^2} \\ \frac{2}{|i_2|^2}+\frac{2}{|i_1|^2} & \frac{1}{|i_2|^2} & \cdots & \frac{2}{|i_2|^2}+\frac{2}{|i_b|^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{|i_b|^2}+\frac{2}{|i_1|^2} & \frac{2}{|i_b|^2}+\frac{2}{|i_2|^2} & \cdots & \frac{1}{|i_b|^2}\end{pmatrix}_{b\times b} = A,$$

$$\frac{\partial\omega_1}{\partial\xi_2} = \frac{\partial\omega_2}{\partial\xi_1} = 0_{b\times b}.$$

Thanks to Sec. 3.2 in Ref. 37, $\det\left(\frac{\partial\omega_1}{\partial\xi_1}\right) = \det\left(\frac{\partial\omega_2}{\partial\xi_2}\right) \neq 0$. It is easy to check that $\det\left(\frac{\partial\omega}{\partial\xi}\right) = \det\left(\frac{\partial\omega_1}{\partial\xi_1}\right)\cdot\det\left(\frac{\partial\omega_2}{\partial\xi_2}\right) \neq 0$.

Thus $(A1)$ is verified

Verificatio of $(A2)$ *and* $(A2^*)$: Take $a = 4$, we can move ξ to get

$$|\Omega^0_{1n} - \Omega^0_{2n}| = \frac{1}{2\pi^2}\sum_{i\in\mathcal{I}}\left(\frac{1}{|i|^2}+\frac{1}{|n|^2}\right)(\xi_{1i}-\xi_{2i}) > \gamma, n\in\mathbb{Z}^2_{odd,1}.$$

Thus $(A2)$ and $(A2^*)$ are verified

Verificatio of $(A3)$: For $h = 1, 2, M_{hn}$ read as follows:

$$M_{hn} = \Omega_{hn},\ n\in\mathbb{Z}^2_{odd,1}\backslash(\mathcal{L}_1\cup\mathcal{L}_2),$$

$$M_{hn} = \begin{pmatrix}\Omega_{hn}+\omega_{hi} & Q_n \\ Q_m & \Omega_{hm}+\omega_{hj}\end{pmatrix}\ n\in\mathcal{L}_1,$$

$$M_{hn} = \begin{pmatrix}\Omega_{hn}-\omega_{hi} & -\tilde{Q}_n \\ \tilde{Q}_m & -(\Omega_{hm}-\omega_{hj})\end{pmatrix}\ n\in\mathcal{L}_2.$$

This part is the same as Ref. 37. In the following, we only give the proof for the case: $n\in\mathcal{L}_1, n'\in\mathcal{L}_2$

$$\langle k,\omega\rangle I_4 \pm M_{1n}\otimes I_2 \pm I_2\otimes M_{2n'}.$$

Set $\alpha = \varepsilon^{-4}(|i_1|^2, |i_2|^2, \ldots, |i_b|^2), \xi_1 = (\xi_{1i_1}, \xi_{1i_2}, \ldots, \xi_{1i_b}), \xi_2 = (\xi_{2i_1}, \xi_{2i_2}, \ldots, \xi_{2i_b})$,

$$v_{ab} = \frac{1}{2\pi^2}\left(\frac{1}{|a|^2} + \frac{1}{|b|^2} + \frac{2}{|i_1|^2}, \frac{1}{|a|^2} + \frac{1}{|b|^2} + \frac{2}{|i_2|^2}, \ldots, \frac{1}{|a|^2} + \frac{1}{|b|^2} + \frac{2}{|i_b|^2}\right),$$

$$\bar{v}_{ab} = \frac{1}{2\pi^2}\left(\frac{1}{|a|^2} - \frac{1}{|b|^2}\right)(1, 1, \ldots, 1),$$

with $a \in \{n, m, n', m'\}$ and $b \in \{i, j, i', j'\}$.

Its eigenvalues are

$$2\langle k_1, \alpha\rangle \pm \varepsilon^{-4}(|n|^2 + |i|^2) \pm \varepsilon^{-4}(|n'|^2 - |i'|^2) + \langle Ak \pm v_{ni}, \xi_1\rangle + \langle Ak \pm \bar{v}_{n'i'}, \xi_2\rangle$$
$$\pm\left[\frac{1}{8\pi^2|i|^2}\xi_{1i} + \frac{1}{8\pi^2|j|^2}\xi_{1j} \pm \frac{\sqrt{\Delta}}{2}\right] \pm \left[\frac{1}{8\pi^2|i'|^2}\xi_{2i'} + \frac{1}{8\pi^2|j'|^2}\xi_{2j'} \pm \frac{\sqrt{\bar{\Delta}'}}{2}\right],$$

where

$$\Delta = H_i^2\xi_{1i}^2 + H_j^2\xi_{1j}^2 + (-2H_iH_j + 4Q_nQ_m)\xi_{1i}\xi_{1j},$$

$$\bar{\Delta}' = H_{i'}^2\xi_{2i'}^2 + H_{j'}^2\xi_{2j'}^2 + (2H_{i'}H_{j'} - 4\bar{Q}_{n'}\bar{Q}_{m'})\xi_{2i'}\xi_{2j'},$$

with $H_i = \frac{1}{4\pi^2|i|^2}, H_j = \frac{1}{4\pi^2|j|^2}$. Hence all the eigenvalues are not identically zero due to the presence of the square root terms thanks to the difference between ξ_1 and ξ_2.

$\langle k, \omega\rangle I_4 \pm M_{1n} \otimes I_2 \pm I_2 \otimes M_{2n'}$ is polynomizl function in ξ of order at most four. Thus

$$|\partial_\xi^4 \det(\langle k, \omega\rangle I_4 \pm M_{1n} \otimes I_2 \pm I_2 \otimes M_{2n'})| \geq \frac{1}{2}|k| \neq 0.$$

By excluding some parameter set with measure $O(\gamma^{\frac{1}{4}})$, we have

$$|\det(\langle k, \omega\rangle I_4 \pm M_{1n} \otimes I_2 \pm I_2 \otimes M_{2n'})| \geq \frac{\gamma}{|k|^\tau}, k \neq 0, n \in \mathcal{L}_1, n \in \mathcal{L}_2.$$

The cases $k = 0, h \neq h', n = n'$ in (2.10) can be verifie by $(A2^*)$. We consider one case for $n \in \mathcal{L}_1$,

$$\operatorname{diag}\begin{pmatrix}\Omega_{1n} + \omega_{1i} - \Omega_{2n} - \omega_{2i}\\ \Omega_{1n} + \omega_{1i} - \Omega_{2m} - \omega_{2j}\\ \Omega_{1m} + \omega_{1j} - \Omega_{2n} - \omega_{2i}\\ \Omega_{1m} + \omega_{1j} - \Omega_{2m} - \omega_{2j}\end{pmatrix} + \begin{pmatrix}0 & -Q_n & Q_n & 0\\ -Q_m & 0 & 0 & Q_n\\ Q_m & 0 & 0 & -Q_n\\ 0 & Q_m & -Q_m & 0\end{pmatrix}.$$

Although $|i|^2 + |n|^2 - |m|^2 - |j|^2 = 0$, the diagonal part still has

$$\frac{1}{2\pi^2}\left(\operatorname{diag}\begin{pmatrix}\sum_{\tilde{i}\in\mathcal{I}}\left(\frac{1}{|\tilde{i}|^2} + \frac{1}{|n|^2}\right)(\xi_{1\tilde{i}} - \xi_{2\tilde{i}})\\ \sum_{\tilde{i}\in\mathcal{I}}\left(\frac{1}{|\tilde{i}|^2} + \frac{1}{|n|^2}\right)\xi_{1\tilde{i}} - \left(\frac{1}{|\tilde{i}|^2} + \frac{1}{|m|^2}\right)\xi_{2\tilde{i}}\\ \sum_{\tilde{i}\in\mathcal{I}}\left(\frac{1}{|\tilde{i}|^2} + \frac{1}{|m|^2}\right)\xi_{1\tilde{i}} - \left(\frac{1}{|\tilde{i}|^2} + \frac{1}{|n|^2}\right)\xi_{2\tilde{i}}\\ \sum_{\tilde{i}\in\mathcal{I}}\left(\frac{1}{|\tilde{i}|^2} + \frac{1}{|m|^2}\right)(\xi_{1\tilde{i}} - \xi_{2\tilde{i}})\end{pmatrix},\right.$$
$$\left.+\operatorname{diag}\begin{pmatrix}\frac{1}{2|i|^2}(\xi_{1i} - \xi_{2i}) + \sum_{\tilde{i}\in\mathcal{I},\tilde{i}\neq i}\left(\frac{1}{|i|^2} + \frac{1}{|\tilde{i}|^2}\right)(\xi_{1\tilde{i}} - \xi_{2\tilde{i}})\\ \frac{1}{2}\left(\frac{\xi_{1i}}{|i|^2} - \frac{\xi_{2j}}{|j|^2}\right) + \sum_{\tilde{i}\in\mathcal{I},\tilde{i}\neq i}\left(\frac{\xi_{1\tilde{i}}}{|i|^2} + \frac{\xi_{1\tilde{i}}}{|\tilde{i}|^2}\right) - \sum_{\tilde{i}\in\mathcal{I},\tilde{i}\neq j}\left(\frac{\xi_{2\tilde{i}}}{|j|^2} + \frac{\xi_{2\tilde{i}}}{|\tilde{i}|^2}\right)\\ \frac{1}{2}\left(\frac{\xi_{1j}}{|j|^2} - \frac{\xi_{2i}}{|i|^2}\right) + \sum_{\tilde{i}\in\mathcal{I},\tilde{i}\neq j}\left(\frac{\xi_{1\tilde{j}}}{|j|^2} + \frac{\xi_{1\tilde{i}}}{|\tilde{i}|^2}\right) - \sum_{\tilde{i}\in\mathcal{I},\tilde{i}\neq i}\left(\frac{\xi_{2\tilde{i}}}{|i|^2} + \frac{\xi_{2\tilde{i}}}{|\tilde{i}|^2}\right)\\ \frac{1}{2|j|^2}(\xi_{1j} - \xi_{2j}) + \sum_{\tilde{i}\in\mathcal{I},\tilde{i}\neq j}\left(\frac{1}{|j|^2} + \frac{1}{|\tilde{i}|^2}\right)(\xi_{1\tilde{i}} - \xi_{2\tilde{i}})\end{pmatrix}\right)$$

we can move ξ to make

$$\left|\det\left(\operatorname{diag}\begin{pmatrix}\Omega_{1n}+\omega_{1i}-\Omega_{2n}-\omega_{2i}\\ \Omega_{1n}+\omega_{1i}-\Omega_{2m}-\omega_{2j}\\ \Omega_{1m}+\omega_{1j}-\Omega_{2n}-\omega_{2i}\\ \Omega_{1m}+\omega_{1j}-\Omega_{2m}-\omega_{2j}\end{pmatrix}+\begin{pmatrix}0 & -Q_n & Q_n & 0\\ -Q_m & 0 & 0 & Q_n\\ Q_m & 0 & 0 & -Q_n\\ 0 & Q_m & -Q_m & 0\end{pmatrix}\right)\right|\geq\gamma.$$

In other cases, the proof is similar to Sec. 3.2 in Ref. 37, so we omit it. Thus $(A3)$ is verified

Verificatio of $(A4)$: For a given $0<r<1$ and $s=\varepsilon^{\frac{1}{2}}$, according to Lemma 3.1, $\|G_{\bar{q}}\|_\rho\leq c\|q\|_\rho^3$, then

$$\sum_{n\in\mathbb{Z}^2_{odd,1}}\|P_{z_n}\|_{\mathcal{O}}e^{|n|\rho}+\sum_{n\in\mathbb{Z}^2_{odd,1}}\|P_{\bar{z}_n}\|_{\mathcal{O}}e^{|n|\rho}=\|P_z\|_\rho+\|P_{\bar{z}}\|_\rho\leq c\|q\|_\rho^3\leq c\left(|I|^{\frac{3}{2}}+\|z\|_\rho^3\right).$$

In addition,

$$\sup_{\|q\|_\rho<2s}\|G\|_{\mathcal{O}}\leq c\sup_{\|q\|_\rho<2s}\|q\|_\rho^4\leq cs^4,$$

thus

$$\|P\|_{D_\rho(2r,2s),\mathcal{O}}=\sup_{D_\rho(2r,2s)}\|P\|_{\mathcal{O}}\leq cs^4.$$

According to Cauchy estimates,

$$\|P_I\|_{D_\rho(r,s),\mathcal{O}}\leq cs^2,\ \|P_\theta\|_{D_\rho(r,s),\mathcal{O}}\leq cs^4,$$

then

$$\begin{aligned}\|X_P\|_{D_\rho(r,s),\mathcal{O}}&=\|P_I\|_{D_\rho(r,s),\mathcal{O}}+\frac{1}{s^2}\|P_\theta\|_{D_\rho(r,s),\mathcal{O}}\\&\quad+\sup_{D_\rho(r,s)}\left[\frac{1}{s}\sum_{n\in\mathbb{Z}^2_{odd,1}}\|P_{z_n}\|_{\mathcal{O}}e^{|n|\rho}+\frac{1}{s}\sum_{n\in\mathbb{Z}^2_{odd,1}}\|P_{\bar{z}_n}\|_{\mathcal{O}}e^{|n|\rho}\right]\\&\leq cs^2+\frac{cs^4}{s^2}+c\sup_{D_\rho(r,s)}\frac{1}{s}\left(|I|^{\frac{3}{2}}+\|z\|_\rho^3\right)\\&\leq cs^2\leq c\varepsilon.\end{aligned}$$

Thus $(A4)$ is verified

Verificatio of $(A5)$:

$$\begin{aligned}P=&\sum_{\sum_{h=1}^{2}\left(\sum_{j=1}^{b}k_{hj}i_j+\sum_{n\in\mathbb{Z}^2_{odd,1}}(\alpha_{hn}-\beta_{hn})n\right)=0}P_{\alpha,\beta}\sqrt{I_{1,i_1}+\xi_{1,i_1}}^{\alpha_{1,i_1}+\beta_{1,i_1}}\cdots\\&\sqrt{I_{1,i_b}+\xi_{1,i_b}}^{\alpha_{1,i_b}+\beta_{1,i_b}}\sqrt{I_{2,i_1}+\xi_{2,i_1}}^{\alpha_{2,i_1}+\beta_{2,i_1}}\cdots\sqrt{I_{2,i_b}+\xi_{2,i_b}}^{\alpha_{2,i_b}+\beta_{2,i_b}}\\&\times e^{i\sum_{h=1}^{2}\sum_{j=1}^{b}(\alpha_{h,i_j}-\beta_{h,i_j})\theta_{h,j}}z^{\alpha-\sum_{h=1}^{2}\sum_{j=1}^{b}\alpha_{h,i_j}e_{h,i_j}}\bar{z}^{\beta-\sum_{h=1}^{2}\sum_{j=1}^{b}\beta_{h,i_j}e_{h,i_j}}.\end{aligned}$$

Let $k=(k_{1,1},\ldots,k_{2,b})=(\alpha_{1,i_1}-\beta_{1,i_1},\ldots,\alpha_{2,i_b}-\beta_{2,i_b})$,

$$P=\sum_{\sum_{h=1}^{2}\left(\sum_{j=1}^{b}k_{hj}i_j+\sum_{n\in\mathbb{Z}^2_{odd,1}}(\alpha_{hn}-\beta_{hn})n\right)=0}P_{kl\alpha\beta}(\xi)e^{i\langle k,\theta\rangle}I^l z^\alpha\bar{z}^\beta,$$

where $k=(k_1,k_2),k_1\in\mathbb{Z}^b,k_2\in\mathbb{Z}^b;l\in\mathbb{N}^{2b},\alpha,\beta\in\mathbb{N}^{2\mathbb{Z}^2_{odd,1}}$ has the following relation

$$\sum_{h=1}^{2}\left(\sum_{j=1}^{b}k_{hj}i_j+\sum_{n\in\mathbb{Z}^2_{odd,1}}(\alpha_{hn}-\beta_{hn})n\right)=0.$$

Then $(A5)$ is verified

Verificatio of $(A6)$: We only need to check F satisfie $(A6)$, then refer to Lemma 4.4 in Ref. 15, the X_F^1 preserves $(A6)$ property of $G, \tilde{G}$. Recall that (3.2), F is given as

$$F = \sum_{S_1,h} \frac{i\varepsilon}{\lambda_i + \lambda_j + \lambda_n + \lambda_m} g_{h,i,j,n,m}\big(q_{hi}q_{hj}q_{hn}q_{hm} + \bar{q}_{hi}\bar{q}_{hj}\bar{q}_{hn}\bar{q}_{hm}\big)$$
$$+ \sum_{S_2,h} \frac{i\varepsilon}{\lambda_i + \lambda_j - \lambda_n + \lambda_m} g_{h,i,j,-n,m}\big(q_{hi}q_{hj}\bar{q}_{hn}q_{hm} + \bar{q}_{hi}\bar{q}_{hj}q_{hn}\bar{q}_{hm}\big)$$
$$+ \sum_{S_3,h} \frac{i\varepsilon}{\lambda_i - \lambda_j + \lambda_n - \lambda_m} g_{h,i,-j,n,-m}\big(q_{hi}\bar{q}_{hj}q_{hn}\bar{q}_{hm} + \bar{q}_{hi}q_{hj}\bar{q}_{hn}q_{hm}\big).$$

Then for t large enough and $\forall c \in \mathbb{Z}^2_{odd}\backslash\{0\}$, we have

$$\sum_{i,j,n,m,t,h} \frac{i\varepsilon}{\lambda_i - \lambda_j + \lambda_{n+tc} - \lambda_{m+tc}} g_{h,i,-j,n+tc,-(m+tc)} q_{hi}\bar{q}_{hj}q_{h,n+tc}\bar{q}_{h,m+tc}$$
$$= \sum_{i,j,n,m,t,h} \frac{i\varepsilon}{|i|^2 - |j|^2 + |n|^2 - |m|^2 + 2t\langle n-m, c\rangle} g_{h,i,-j,n+tc,-(m+tc)} q_{hi}\bar{q}_{hj}q_{h,n+tc}\bar{q}_{h,m+tc}.$$

According to the mathematical analysis, we obtain the limits

$$\lim_{t\to\infty} g_{h,i,-j,n+tc,-(m+tc)} = \frac{\langle 4m + 4n - i - j, c\rangle}{2\sqrt{\langle m, c\rangle}\sqrt{\langle n, c\rangle}}.$$

Hence, when $\langle n-m, c\rangle = 0$, $\lim\limits_{t\to\infty} \frac{\partial^2 F}{\partial z_{n+tc}\partial \bar{z}_{m+tc}}$ exists and

$$\Big\| \frac{\partial^2 F}{\partial z_{n+tc}\partial \bar{z}_{m+tc}} - \lim_{t\to\infty} \frac{\partial^2 F}{\partial z_{n+tc}\partial \bar{z}_{m+tc}} \Big\| \le \frac{\varepsilon}{t} e^{-|n-m|\rho}.$$

when $\langle n-m, c\rangle \neq 0$, $\lim\limits_{t\to\infty} \frac{\partial^2 F}{\partial z_{n+tc}\partial \bar{z}_{m+tc}} = 0$ and

$$\Big\| \frac{\partial^2 F}{\partial z_{n+tc}\partial \bar{z}_{m+tc}} - 0 \Big\| \le \frac{\varepsilon}{t} e^{-|n-m|\rho}.$$

Similarly,

$$\Big\| \frac{\partial^2 F}{\partial z_{n+tc}\partial z_{m-tc}} - \lim_{t\to\infty} \frac{\partial^2 F}{\partial z_{n+tc}\partial z_{m-tc}} \Big\|, \Big\| \frac{\partial^2 F}{\partial \bar{z}_{n+tc}\partial \bar{z}_{m-tc}} - \lim_{t\to\infty} \frac{\partial^2 F}{\partial \bar{z}_{n+tc}\partial \bar{z}_{m-tc}} \Big\| \le \frac{\varepsilon}{t} e^{-|n+m|\rho}.$$

Thus $(A6)$ is verified

So we have verifie all the assumptions of Theorem 2. By applying Theorem 2, we get Theorem 1.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflict to disclose.

Author Contributions

Shuaishuai Xue: Writing – original draft (equal); Writing – review & editing (equal). **Yingnan Sun**: Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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